High Dimensional Schwartz KP Equations

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The usual (2+1)-dimensional Schwartz Kadomtsev-Petviashvili (KP) equation is extended to $(n+1)$-dimensions. The extension is Painlevé integrable in the sense that it possesses the Painlevé property. A (3+1)-dimensional special one is just found when we solve a real (3+1)-dimensional KP equation approximately.

Key words: High Dimensional Integrable Models; Schwartz KP Equation; Painlevé Property.

To find high dimensional integrable models is one of the important problems in mathematical physics [1, 2]. Recently, we have established some possible ways to find high dimensional integrable models under some special conditions. (i) According to the fact that all the known (2+1)-dimensional integrable models possess a common generalized Virasoro type symmetry algebra [3], a general method to get some integrable models under the condition that they possess the generalized Virasoro symmetry algebra [4 - 6] is proposed; (ii) It is also known that every (1+1)- and (2+1)-dimensional integrable model possesses a Schwartz form which is conformal invariant and the conformal invariance plays a very important role to find other integrable properties [7]. Basing on this fact, we point out that starting from a conformal invariant form is one of the most convenient ways to get higher dimensional integrable models [8]; (iii) After embedding the lower dimensional integrable models in higher dimensions and extending the Painlevé analysis approach to a new form, we can systematically obtain many higher dimensional Painlevé integrable models from lower dimensional ones [9]; (iv) Using some noninvertible Miura type transformations, we established another way to find some nontrivial higher dimensional integrable models from trivial integrable ones [10]; (v) Starting from a recursion operator of any (1+1)-dimensional integrable model, one can established some integrable breaking soliton equations in any dimension; (vi) Using inner parameter dependent symmetry constraints to the lower dimensional integrable models, one may also obtain some higher dimensional integrable models [11].

From [7], we see that the conformal invariance of the Schwartz form may be an intrinsic property in the integrable models. For instance, the infinitely many symmetries of the Korteweg-de Vries (KdV) equation result from nothing but the conformal invariance in the solution space $\mathcal{S} \equiv \{\phi_i,\ i = 1,\ 2,\ \dots\infty\}$ of the Schwartz KdV equations

$$\frac{\phi_{i,t}}{\phi_{i,x}} + \{\phi_i;\ x\} + \lambda_i = 0,\ i = 1,\ 2, \dots, \infty \quad (1)$$

with $\{\phi_i;\ x\} \equiv \left(\phi_{i,xx}/\phi_{i,x}\right)_x - \frac{1}{2}\left(\phi_{i,xx}/\phi_{i,x}\right)^2$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. This fact shows that, if we want to construct some integrable models, conformal invariant forms may be best the candidates. According to this idea, we have prove that a quite general extention of the Schwartz KdV equation in any dimension is Painlevé integrable [8]. Naturally, a further question should be answered: Whether the other known lower dimensional integrable Schwartz equations also be extended to higher dimensional Painlevé integrable ones?

In this short paper we extended the Schwartz KP equation [12]

$$\left(\frac{\phi_t}{\phi_x} + \{\phi;\ x\} + \frac{3}{2}\frac{\phi_y^2}{\phi_x^2}\right)_x + 3\left(\frac{\phi_y}{\phi_x}\right)_y = 0 \quad (2)$$

to high dimensions. The Schwartz KP equation (2) is

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related to the usual KP equation [13]

$$(u_t + u_{xxx} - 6uu_x)_x + 3u_{yy} = 0 \qquad (3)$$

by [7, 14]

$$u = \frac{1}{2}\{\phi;\, x\} - \frac{\phi_{xx}^2}{2\phi_x^2} - \frac{\phi_y^2}{4\phi_x^2} + \frac{1}{2}\int^x \left(\frac{\phi_y}{\phi_x}\right)_y \mathrm{d}x. \qquad (4)$$

The KP equation (3) has been studied deeply by many authors [15 - 20] because of its wide applications in physics (say, the surface wave and internal waves in straits, channels or oceans of varying depth and width [13, 15, 1]) and elegant mathematical structure.

To extend the Schwartz KP equation (2) in high dimensions, we may take many forms. Here we write down an $(n+1)$-dimensional simplest form

$$\sum_{i=1}^{n}\left(a_i\frac{\phi_t}{\phi_{x_i}} + b_i\{\phi;\, x_i\} + \frac{3}{2}\sum_{j=1}^{n} a_{ij}\frac{\phi_{x_j}^2}{\phi_{x_i}^2}\right)_{x_i} \qquad (5)$$

$$+\, 3b_{ij}\sum_{j=1}^{n}\left(\frac{\phi_{x_j}}{\phi_{x_i}}\right)_{x_j} = 0,$$

where $a_i,\ b_i,\ a_{ij}$, and b_{ij} are arbitrary constants. It is obviously that (5) is conformal invariant, i.e., (5) is invariant under the Möbious transformation

$$\phi \longrightarrow \frac{a + b\phi}{c + d\phi},\ ad \neq bc. \qquad (6)$$

To check the integrability of a model, the Painlevé analysis formulated by Weiss, Tabor and Carnevale (WTC) [21] is one of the most powerful methods. In order to make use of the WTC approach, we make, as in [8, 9], the transformations

$$\phi = \exp F, \qquad (7)$$

and

$$u_0 = F_t,\ u_i = F_{x_i},\ i = 1,\ 2,\ \dots,\ n \qquad (8)$$

at first. Substituting (7) and (8) in (5), the $(n+1)$-dimensional Schwartz KP equation (5) becomes an equation system

$$\sum_{i=1}^{n}\left(a_i\frac{u_0}{u_i} + b_i\left(\frac{u_{i,x_ix_i}}{u_i} - \frac{1}{2}u_i^2 - \frac{3u_{i,x_i}^2}{2u_i^2}\right) + \sum_{j=1}^{n}\frac{3}{2}a_{ij}\frac{u_j^2}{u_i^2}\right)_{x_i}$$

$$+\, 3b_{ij}\sum_{j=1}^{n}\left(\frac{u_j}{u_i}\right)_{x_j} = 0, \qquad (9)$$

$$u_{i,t} = u_{0,x_i},\ i = 1,\ 2,\ \dots,\ n, \qquad (10)$$

where the equations (10) come from the compatibility conditions of the transformations (8).

Now using, the standard WTC approach, we can prove that the equation system (9, 10) possesses the Painlevé property. According to the WTC approach, we state that $(n+1)$ dimensional model possesses the Painlevé property if its solutions are single-valued about an arbitrary singularity manifold which is given by $\phi_1(x_1,\ x_2,\ \dots,\ x_n,\ t) = 0$.

With help of the leading order analysis we know that the functions $\{u_i,\ i = 0,\ 1,\ 2,\ \dots,\ n\}$ should be expanded as

$$u_i = \sum_{m=0}^{\infty} u_{im}\phi_1^{m-1},\ i = 0,\ 1,\ 2,\ \dots,\ n \qquad (11)$$

with

$$u_{00}^2 = \phi_{1,t}^2,\ u_{i0} = u_{00}\phi_{1,x_i}/\phi_{1,t}\,. \qquad (12)$$

Substituting (11) into (9) and (10) and using (12) we have

$$(m+1)(m-1)(m-2)\sum_{i=0}^{n} b_i u_{i0}^2 u_{im} \qquad (13)$$

$$= f(u_{ik},\ i = 0,\ 1,\ 2,\ \dots,\ n,\ k \leq m-1),$$

$$(m-1)(u_{0m}u_{i0} - u_{im}u_{00}) = u_{i(j-1),t} - u_{0(j-1),x_i}, \qquad (14)$$

where f is a complicated function of $\{u_{ik}, i = 0, 1, 2, \dots, n, k \leq m-1\}$ and the derivatives of the singularity manifold ϕ_1. From (13) and (14), it is not difficult to see that the resonance points are located at

$$m = -1,\ \underbrace{1,\ 1,\ \cdots,\ 1}_{n+1}\ 2\,. \qquad (15)$$

The resonance at $m = -1$ corresponds to the arbitrary singularity manifold ϕ_1. At $n+1$ resonance $m = 1$ and one resonance $m = 2$, there are $n+2$ compatibility conditions

$$\sum_{i=1}^{n} b_i\left(2\phi_{1,x_i}\phi_{1,x_ix_i} - \frac{u_{i0,x_i}}{u_{i0}}\phi_{1,x_i}^2 - u_{i0}u_{i0,x_i}\right) = 0, \qquad (16)$$

$$\sum_{i=1}^{n} b_i\Big(\phi_{1,x_ix_ix_i} + \frac{u^2_{i0,x_i}}{u^2_{i0}}\phi_{1,x_i} - \frac{u_{i0,x_i}}{u_{i0}}\phi_{1,x_ix_i} - \frac{u_{i0,x_ix_i}}{u_{i0}}\phi_{1,x_i}\Big) \quad (17)$$

$$+3\sum_{i,j=1}^{n} b_{ij}\Big(\frac{u_{j0}}{u_{i0}}\phi_{1,x_j} - \frac{u^2_{j0}}{u^2_{i0}}\phi_{1,x_i}\Big) = 0,$$

and

$$u_{i0,t} - u_{00,x_i} = 0,\ i = 1,\ 2, ..., n, \quad (18)$$

which should be satisfied naturally. Fortunately, it is straightforward to see that the conditions (16) - (18) are satisfied identically due to (12). So the system (9, 10) is integrable in the sense that it possesses the Painlevé property. Then the $(n+1)$-dimensional Schwartz KP equation (5) is integrable also.

Using the same approach used above and proposed in [8], we can prove that the further extension of (5) in the form

$$\sum_{i,j=0}^{n} a_{ij}\{\phi;\ x_i\}_{x_j} + \sum_{k=0}^{n}(G_{k,x_k} + H_k) = 0, \quad (19)$$

where a_{ij} are constants and G_k and H_k are polynomial functions of $\{\phi_{x_j}/\phi_{x_i}, (i, j = 0, 1, 2, ..., n)\}$, is also Painlevé integrable (may be changed to a form with the Painlevé property). We omit the details of the proof procedure because of the similarity.

Finally, because the real physical space is (3+1)-dimensional, we list some special simple (3+1)-dimensional examples from (5) and/or (19):

(i)

$$\Big(\frac{\phi_t}{\phi_x} + \{\phi;\ x\} + \frac{3}{4}\frac{\phi_y^2 + \phi_z^2}{\phi_x^2}\Big)_x + \frac{3}{2}\Big(\frac{\phi_y}{\phi_x}\Big)_y + \frac{3}{2}\Big(\frac{\phi_z}{\phi_x}\Big)_z = 0, \quad (20)$$

(ii)

$$\Big(\frac{\phi_t}{\phi_x} + \{\phi;\ x\} + \frac{3}{2}\frac{\phi_y^2}{\phi_z^2}\Big)_x + 3\Big(\frac{\phi_y}{\phi_z}\Big)_y = 0, \quad (21)$$

(iii)

$$\Big(\frac{\phi_t}{\phi_x} + \{\phi;\ x\} + \frac{3}{2}\frac{\phi_y^2}{\phi_x^2}\Big)_z + 3\Big(\frac{\phi_y}{\phi_x}\Big)_y = 0. \quad (22)$$

When $z = x$, the equations (20) - (22) are all reduced back to the usual (2+1)-dimensional Schwartz KP. The first example (20) will be reduced to the Schwartz KP in other two different ways, $z = y$ and $\phi_z = 0$.

It is interesting that the (3+1)-dimentional equation (20) had been proved to be useful to describe the real (3+1)-dimensional physics [22]. Actually, after some suitable approximation, say

$$u = \xi^{-2}\big(u_0 + u_1\xi + u_2\xi^2 + O(\xi^3)\big),\ \xi = \Big(\frac{\phi_x}{\phi} - \frac{\phi_{xx}}{2\phi_x}\Big) \quad (23)$$

with ϕ being giving by (20), one obtaines solutions of the (3+1)-dimensional KP equation [22]

$$(u_t + u_{xxx} - 6uu_x)_x + 3u_{yy} + 3u_{zz} = 0 \quad (24)$$

which describes the dynamics of solitons and nonlinear waves in plasmas and superfluids [23 - 25].

In summary, we have extended the (2+1)-dimensional Schwartz KP equation to arbitrary dimensions. The higher dimensional Schwartz KP equations are integrable in the sence that they can be changed to the forms with Painlevé property. More about the model (especially in the (3+1)-dimensional case) such as multi-soliton solutions, infinitely many conservation laws and symmetries, and other integrable properties as in lower-dimensional cases is worthy of further study.

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